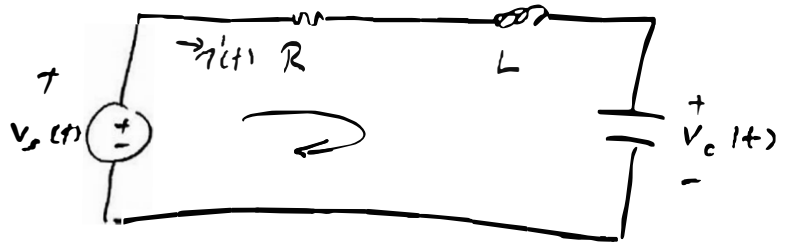
① Find the input/output differential equations when the output is $v_c(t)$



KVL

$$-v_s(t) + R\,i(t) + v_c(t) + L\frac{di(t)}{dt} = 0$$

$v_c(t) = x(t)$ and $v_c(t) = y(t)$

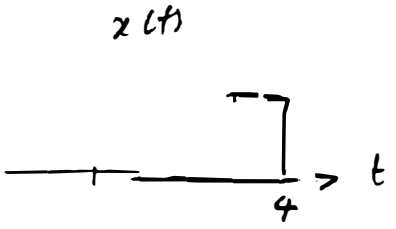
$$i(t) = C\frac{dv_c(t)}{dt} = C\frac{dy(t)}{dt}$$

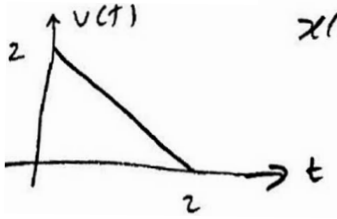
$$-x(t) + RC\frac{dy(t)}{dt} + y(t) + LC\frac{d^2y(t)}{dt} = 0$$

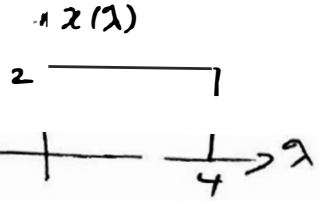
$$\boxed{\frac{d^2y(t)}{dt^2} + \frac{R}{L}\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = \frac{1}{LC}x(t)}$$

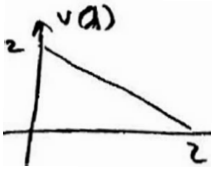
② Compute the convolution $x(t) * v(t)$ for all t.

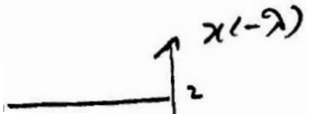
$$x(t) * v(t) = \int_0^t x(t-\lambda)\,v(\lambda)\,d\lambda$$



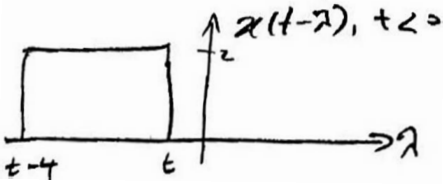


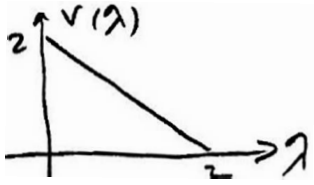


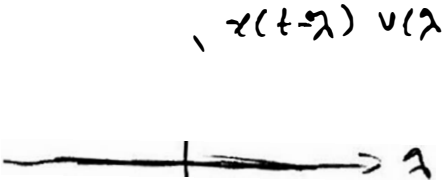




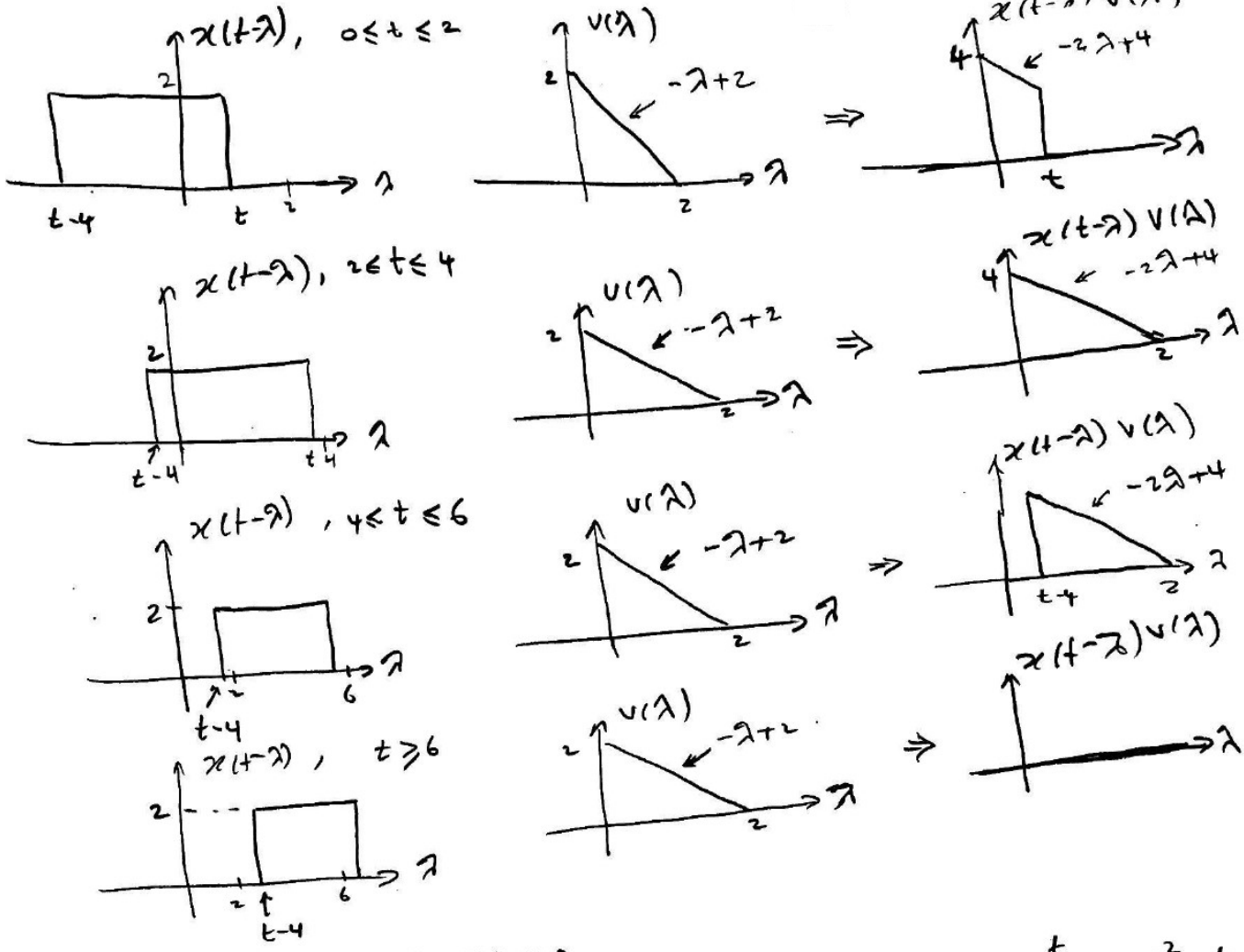
−4







$\Rightarrow \boxed{a = 0}$



For $t<0$, $x(t-\lambda)\,v(\lambda) = 0$

For $0 \le t \le 2$, $x(t-\lambda)\,v(\lambda) = \int_0^t (-2\lambda+4)\,d\lambda = \left[-\lambda^2+4\lambda\right]_0^t = -t^2+4t$

For $2 \le t \le 4$, $x(t-\lambda)\,v(\lambda) = \int_0^2 (-2\lambda+4)\,d\lambda = \left[-\lambda^2+4\lambda\right]_0^2 = 4$

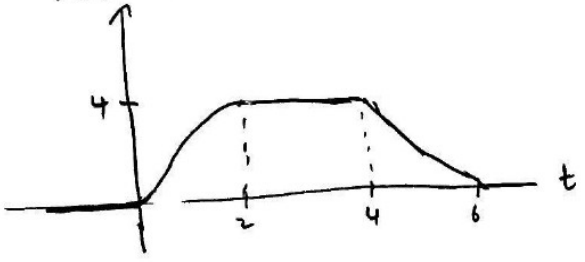
For $4 \le t \le 6$, $x(t-\lambda)\,v(\lambda) = \int_{t-4}^2 (-2\lambda+4)\,d\lambda = \left[-\lambda^2+4\lambda\right]_{t-4}^2$

$= \left[-4+8+(t-4)^2-4(t-4)\right]$

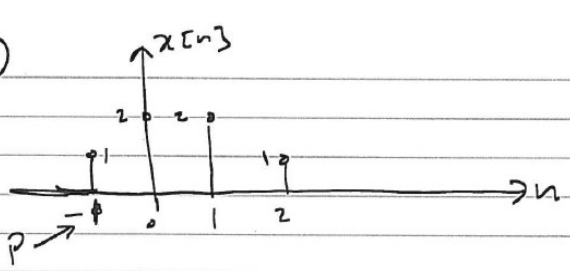
$= \left[4+t^2+16-8t-4t+16\right]$

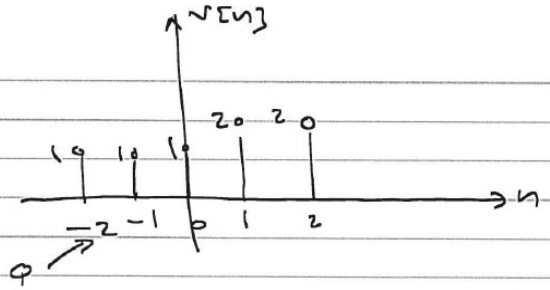
$= t^2-12t+36$

$x(t) * v(t)$



(3)





$y[n] = x[n] * v[n] = ?$

$y[p+q+i] = ?$ $\quad i = 0, 1, 2, \ldots$

$y[-3] = ?\ 1$
$y[-2] = ?\ 3$
$y[-1] = ?\ 5$
$y[0] = ?\ 7$
$y[1] = ?\ 9$
$y[2] = ?\ 9$
$y[3] = ?\ 6$
$y[4] = ?\ 2$

	1	2	2	1
1	1	2	2	1
1	1	2	2	1
1	1	2	2	1
2	2	4	4	2
2	2	4	4	2

$y[n] = 0$ for $n < -3$ & $n > 4$

(4) $y[n+1] + a\,y[n] = x[n]$

$y[n+1] = -a\,y[n] + x[n]$

$\therefore h[n+1] = -a\,h[n] + \delta[n]$

$n=-1 \Rightarrow h[0] = -a\,h[-1] + \delta[-1] = 0$ (both terms 0)

$n=0 \Rightarrow h[1] = -a\,h[0] + \delta[0] = 1$ ($h[0]=0$, $\delta[0]=1$)

$n=1 \Rightarrow h[2] = -a\,h[1] + \delta[1] = -a + 0 = -a$

$n=2 \Rightarrow h[3] = -a\,h[2] + \delta[2] = a^2$

$n=3 \Rightarrow h[4] = -a\,h[3] + \delta[3] = -a^3$

$h[5] = a^4$

$h[6] = -a^5$

$\therefore h[n] = (-a)^{n-1}, \quad n \geq 1$

(5) $x[n] = r[n]$ & $v[n] = u[n]$

$y[n] = x[n] * v[n] = ?$

$y[n] = \sum_{i=0}^{n} r[i]\,u[n-i]$ since $i \leq n \Rightarrow n-i \geq 0 \Rightarrow u[n-i] = 1$

$\therefore y[n] = \sum_{i=0}^{n} r[i] = 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + \cdots$

$n=0 \Rightarrow y[n] = 0$

$n=1 \Rightarrow y[n] = 0 + 1 = 1$

$n=2 \Rightarrow y[n] = 0 + 1 + 2 = 3$

$n=3 \Rightarrow y[n] = 0 + 1 + 2 + 3 = 6$

$n=4 \Rightarrow y[n] = 0 + 1 + 2 + 3 + 4 = 10$

$n=5 \Rightarrow y[n] = 15$

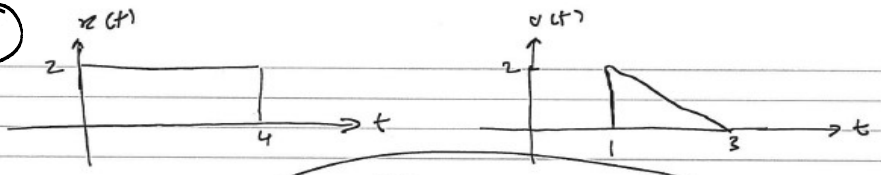
$n=6 \Rightarrow y[n] = 21$

$n=7 \Rightarrow y[n] = 28$

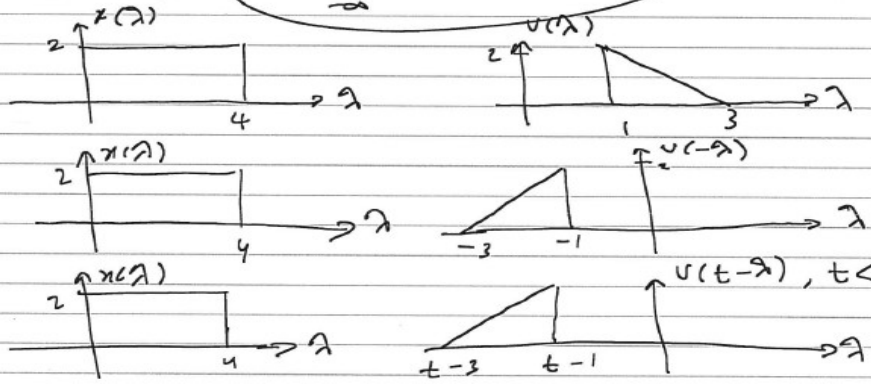
$n=8 \Rightarrow y[n] = 36$

$\therefore y[n] = \frac{n(n+1)}{2}$

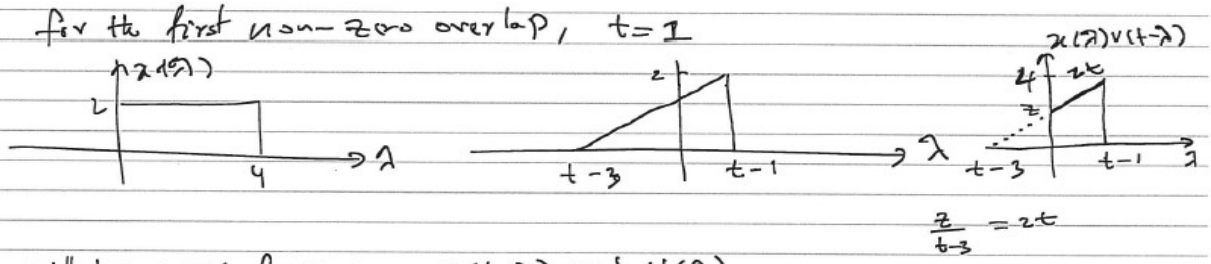
(6)



$$y(t) = x(t) * v(t) = \int_{-\infty}^{\infty} x(\lambda)\, v(t-\lambda)\, d\lambda$$

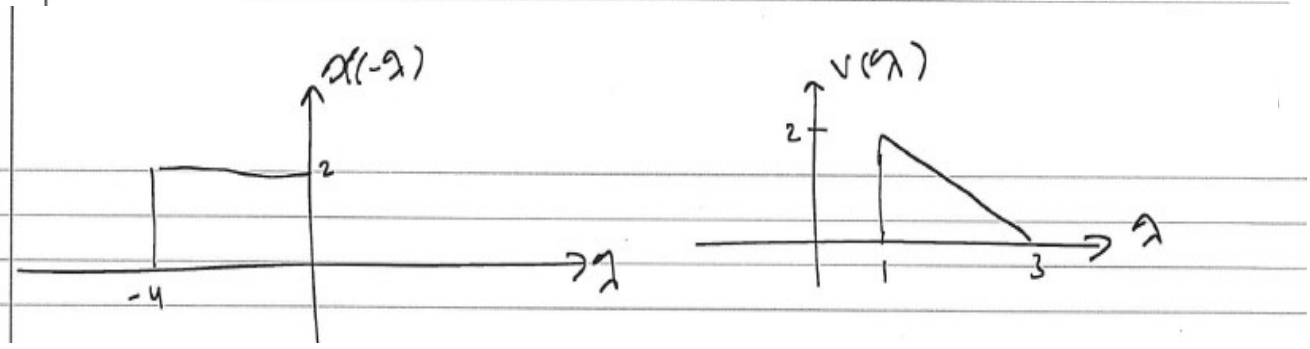


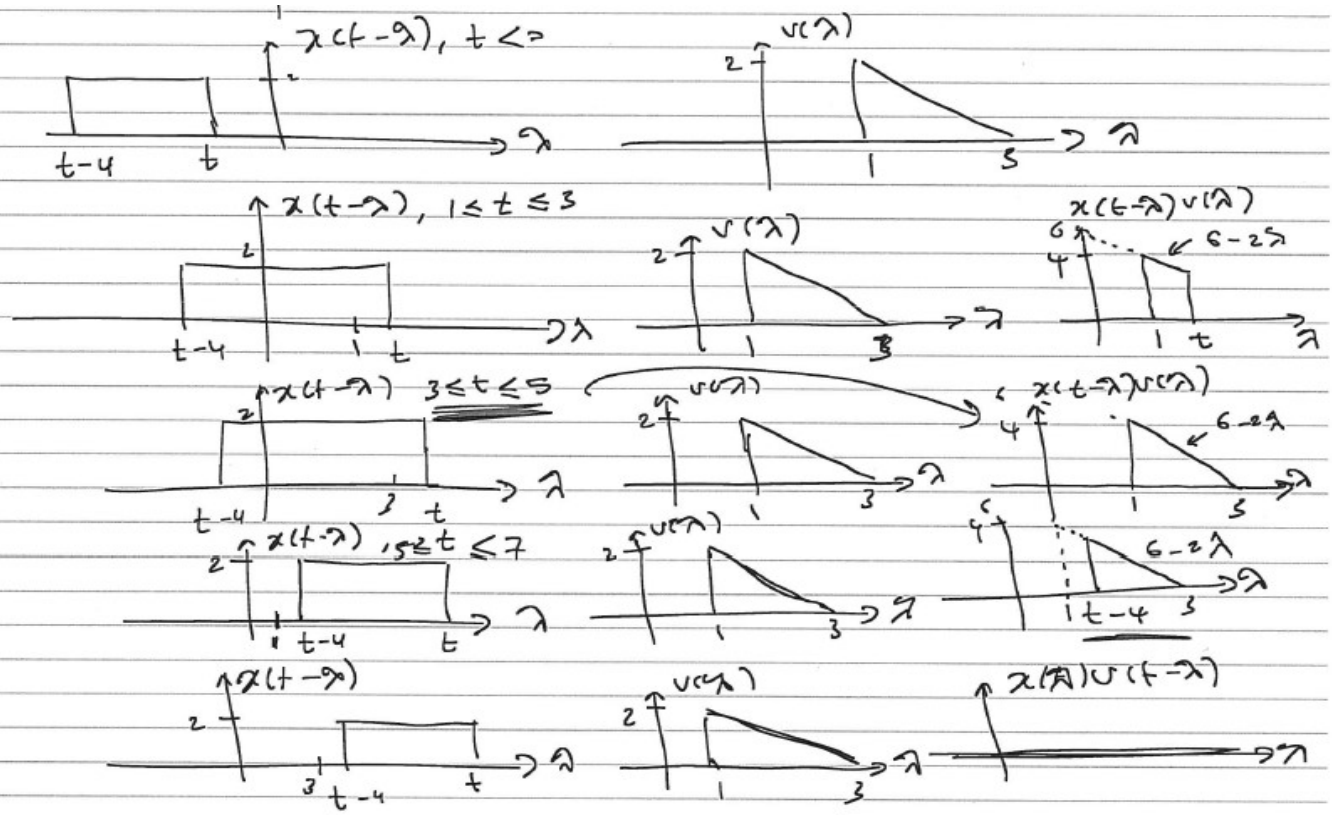
for the first non-zero overlap, $t = 1$



$\frac{2}{t-3} = 2t$

It will be easier if we use $x(t-\lambda)$ and $v(\lambda)$





$t<0 \Rightarrow 0$

$t>7 \Rightarrow 0$

$1 \le t \le 3$

$$x(t)*v(t) = \int_1^t (6-2\lambda)\,d\lambda = 6\lambda - \lambda^2 \Big|_1^t = 6t - t^2 - (6-1) = \underline{6t - t^2 - 5}$$

$3 \le t \le 5$

$$x(t)*v(t) = \int_1^3 (6-2\lambda)\,d\lambda = 6\lambda - \lambda^2 \Big|_1^3 = (18-9) - (6-1)$$

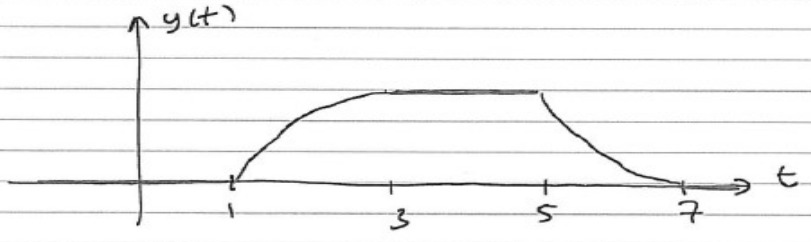
$$= 9 - 5 = \underline{4} \;\ast$$

$5 \le t \le 7$

$$x(t)*v(t) = \int_{t-4}^3 (6-2\lambda)\,d\lambda = 6\lambda - \lambda^2 \Big|_{t-4}^3 = (18-9) - \left(6(t-4) - (t-4)^2\right)$$

$$= 9 - (6t - 24 - t^2 - 16 + 8t)$$

$$= 49 + t^2 - 14t$$



(7)

Show that each of the following continuous-time signals (shown in figure 1 a and b) is equal to the sum of rectangular pulses $P_\tau(t)$ and/or triangular pulses $\left(1-\frac{2|t|}{\tau}\right)P_\tau(t)$. [8 points]

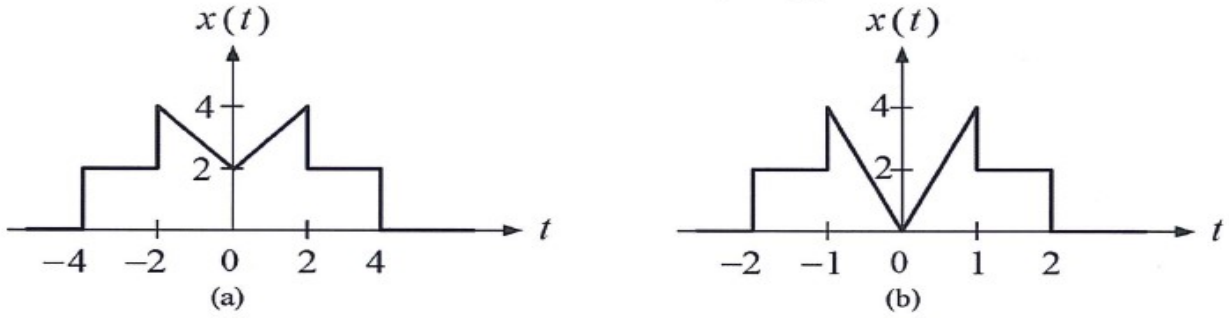


Figure 1. Problem 2 part 1

(a) $2P_8(t) + 2P_4(t) - 2\left(1-\frac{2|t|}{4}\right)P_4(t)$

(b) $2P_4(t) + 2P_2(t) - 4\left(1-\frac{2|t|}{2}\right)P_2(t)$

$$y(t) = \int_0^t \lambda x(\lambda)\, d\lambda$$

$x[n-n_0] \rightarrow 2\cdot x[n-n_0-1]+2$

$y[n-n_0] \rightarrow 2x[n-n_0-1]+2$

(8)

For the following systems, determine whether they are linear, causal, time-invariant, and/or have memories. (answer YES or NO) [16 points]

System	Linear	Causal	Memoryless	Time-invariant
$y(t) = \sin(t)\,x(t)$	Yes	Yes	Yes	No
$y(t) = tx(t+1)$	Yes	No	No	No
$y[n] = 2x[n-1]+2$	No	Yes	No	Yes
$y[n] = n\,x[n]$	Yes	Yes	Yes	No

$a x_1(t) \rightarrow a\sin(t)\,x_1(t) = y_1(t)$

$b x_2(t) \rightarrow b\sin(t)\,x_2(t) = y_2(t)$

$a x_1(t) + b x_2(t) \rightarrow \sin(t)\left(a x_1(t) + b x_2(t)\right) = y_3(t)$

If $y_1(t)+y_2(t) = y_3(t)$ then the system is Linear

$a\sin(t)\,x_1(t) + b\sin(t)\,x_2(t)$

$= \sin(t)\left(a x_1(t) + b x_2(t)\right)$

$x(t-t_0) \rightarrow \sin(t)\,x(t-t_0)$

$y(t-t_0) \rightarrow \sin(t-t_0)\,x(t-t_0)$

(9)

Circle the most correct answer or answers in the multiple-choice problems below (4 points each):

1. For the system shown in Figure 2, the output equal to?

a) $y_1 + y_2$ b) $y_1 * y_2$

c) $y_1/x_1 + y_2/x_2$ d) $y_1x_1 + y_2x_2$

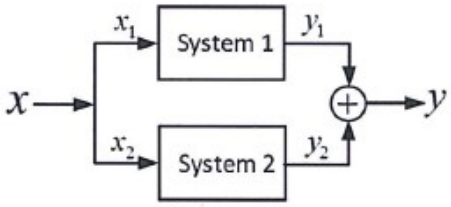


Figure 2.

2. For the system shown in Figure 3, the output equal to?

a) $y_1 - x_2$ b) y_1

c) $y_2 + x - x_1$ d) x_2y_1

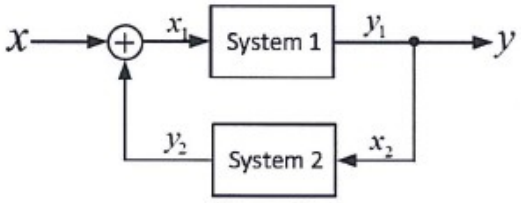


Figure 3.

3. The connection of the system shown in Figure 3 is called:

a) both (b) and (c) are correct b) parallel

c) feedback d) cascade

4. For the signal shown in Figure 4, what is the mathematical expression of this signal?

a) $2P_{12} + 2P_6 + 2(1 - 2|t|/6)P_6(t)$ b) $2P_{12} + 2P_6 + 2(1 - 2|t|/3)P_6(t)$

c) $2P_{12} + 2P_6 - 2(1 - 2|t|/6)P_6(t)$ d) $2P_{12} + 4P_6 + 6(1 - 2|t|/6)P_6(t)$

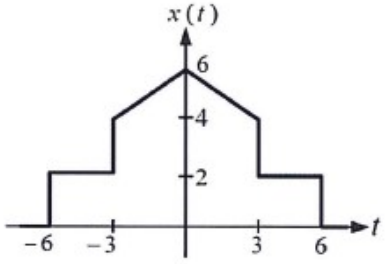


Figure 4.

5. The following system is:

$$y[n] = \sum_{i=0}^{M} b_i x[n-i]$$

a) Recursive b) Non-recursive

(10)

a) For the discrete-time signals $x[n]$ and $v[n]$ shown in figure 5, compute the convolution $x[n] * v[n]$ for all $n \geq 0$ and sketch the results. [12 points]

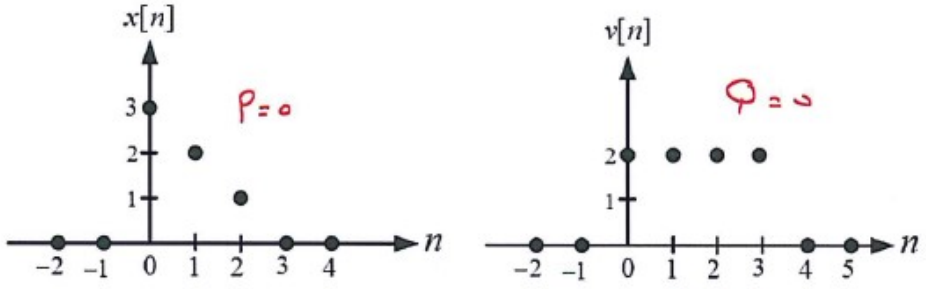
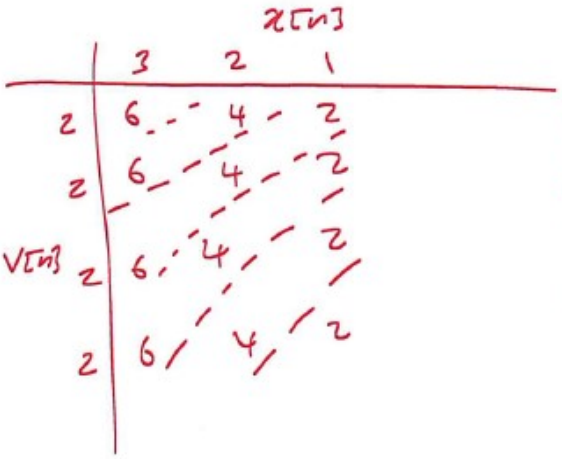
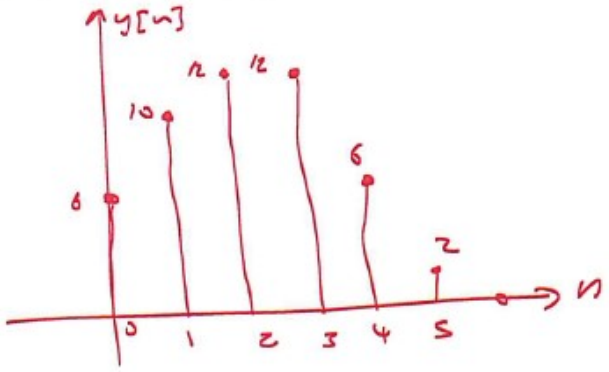


Figure 5.



$y[0] = 6$
$y[1] = 10$
$y[2] = 12$
$y[3] = 12$
$y[4] = 6$
$y[5] = 2$
$y[n] = 0$ for $n \geq 6$



b) For the following discrete-time signals, compute the unit-pulse response $h[n]$ for all integers $n \geq 0$. [8 points]

$$y[n+1] + ay[n] = x[n]$$

$y[n+1] = -a\,y[n] + x[n]$

$\therefore \quad h[n+1] = -a\,h[n] + \delta[n]$

$h[0] = -a\,h[-1] + \delta[-1] = 0$
$h[1] = -a\,h[0] + \delta[0] = 1$
$h[2] = -a\,h[1] + \delta[1] = -a$
$h[3] = -a\,h[2] + \delta[2] = a^2$
$h[4] = -a\,h[3] + \delta[3] = -a^3$

$h[n] = (-a)^{n-1}$, for $n \geq 1$
$h[n] = 0$ for $n = 0$

c) Is the following discrete-time signal periodic? If yes, determine its period. [8 points]

$\Omega_0 = \frac{1}{6}$

$\phi = 0$



Figure 6.

$\Omega_0 N = 2\pi m$

$\Rightarrow \frac{1}{6} N = 2\pi m \Rightarrow N = 12\pi m \Rightarrow$ not a periodic function

11

If the convolution $x[n]*v[n]$ of the given two discrete-time signals $x[n]$ and$v[n]$ is defined by

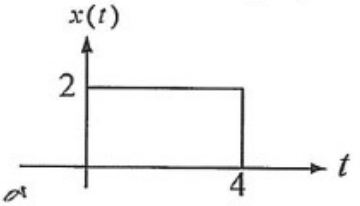
$$x[n] * v[n] = \sum_{i=-\infty}^{\infty} x[i]v[n-i],$$

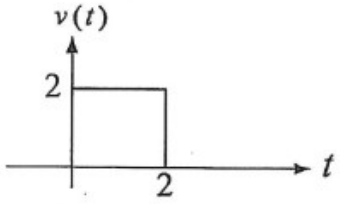
the input/output convolution expression for a ***causal*** system with $x[n] = 0$ for $n < 0$ is defined by:

$\sum_{i=0}^{n} x[i]\,v[n-i]$

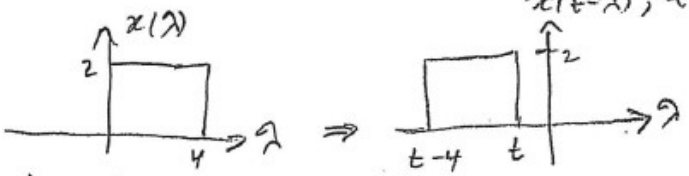
(12)

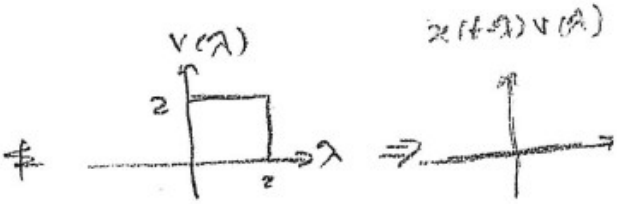
For the continuous-time signals $x(t)$ and $v(t)$ shown in Figure P3E, compute the convolution $x(t) * v(t)$ for all t and sketch your resulting signal. [14 points]





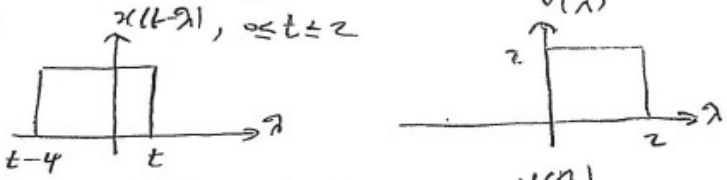
$x(t) * v(t) = \int_{-\infty}^{\infty} x(t-\lambda)\, v(\lambda)\, d\lambda$ Figure P3E

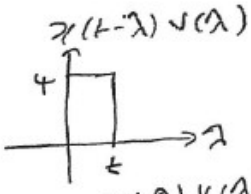




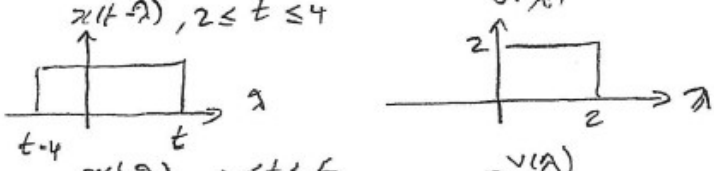
① ∴ for $t<0$, $x(t-\lambda)\,v(\lambda) = 0$

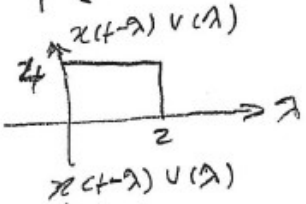
② $x(t-\lambda)$, $0 \le t \le 2$



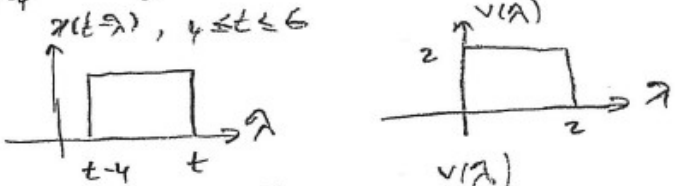


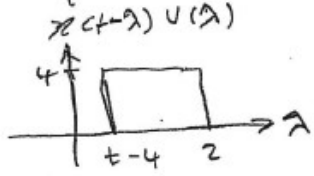
③ $x(t-\lambda)$, $2 \le t \le 4$



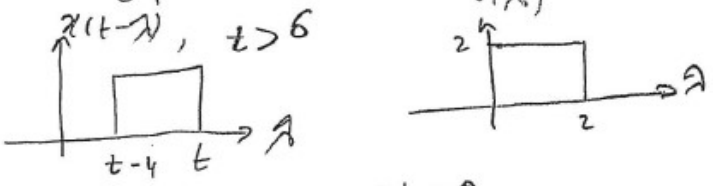


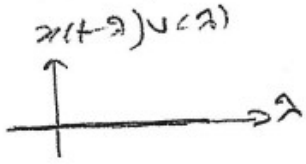
④ $x(t-\lambda)$, $4 \le t \le 6$





⑤ $x(t-\lambda)$, $t > 6$





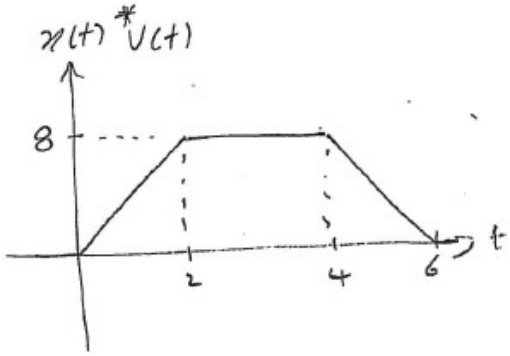
∴ For $t > 6$, $x(t-\lambda)\,v(\lambda) = 0$

For $0 \le t \le 2 \Rightarrow x(t) * v(t) = \int_0^t 4\, d\lambda = 4t$

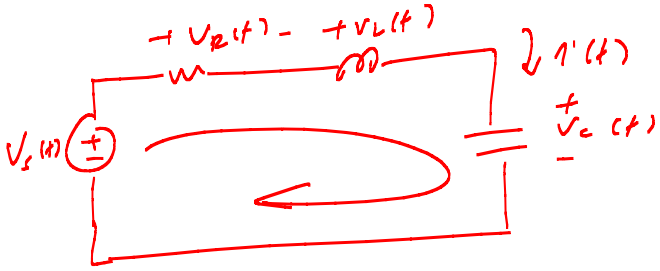
For $2 \le t \le 4 \Rightarrow x(t) * v(t) = \int_0^2 4\, d\lambda = 8$

For $4 \le t \le 6 \Rightarrow x(t) * v(t) = \int_{t-4}^{2} 4\, d\lambda = 4\lambda \Big|_{t-4}^{2}$

$= 8 - 4(t-4) = -4t + 24$



Ex



$x(t) = v_s(t)$

$y(t) = v_R(t)$

$y(t) = R\, i(t) \Rightarrow i(t) = \frac{y(t)}{R}$

$-v_s(t) + v_R(t) + v_L(t) + v_c(t) = 0$

$i(t) = C \frac{dv_c(t)}{dt}$

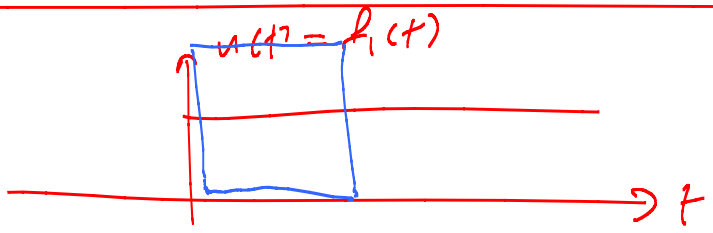
$-x(t) + y(t) + v_L(t) + v_c(t) = 0$

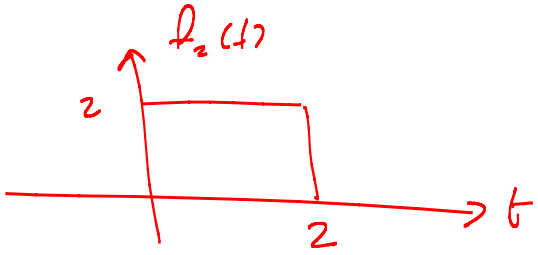
$-x(t) + y(t) + L \frac{di(t)}{dt} + \frac{1}{C} \int i(t)\, dt = 0$

$-\frac{dx(t)}{dt} + \frac{dy(t)}{dt} + L \frac{d^2 i(t)}{dt^2} + \frac{1}{C} i(t) = 0$

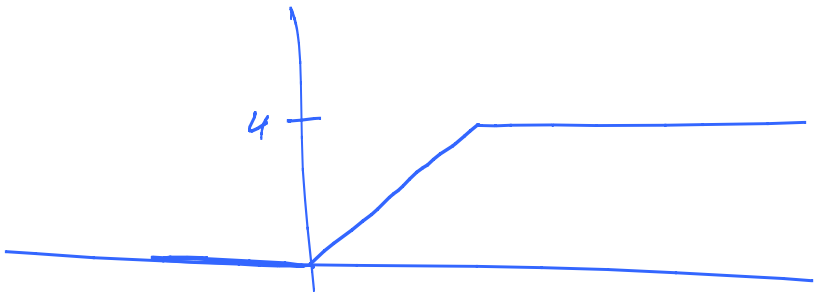
$-\frac{dx(t)}{dt} + \frac{dy(t)}{dt} + \frac{L}{R} \frac{dy(t)}{dt^2} + \frac{1}{RC} y(t) = 0$

$$\frac{d^2y(t)}{dt^2} + \frac{R}{L} \frac{dy(t)}{dt} + \frac{1}{LC} y(t) = \frac{R}{L} \frac{dx(t)}{dt}$$

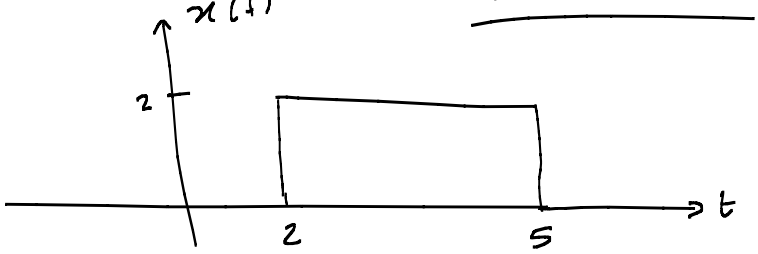


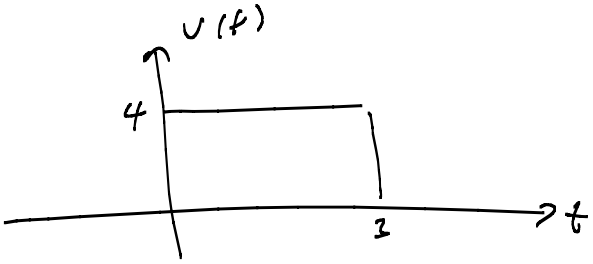


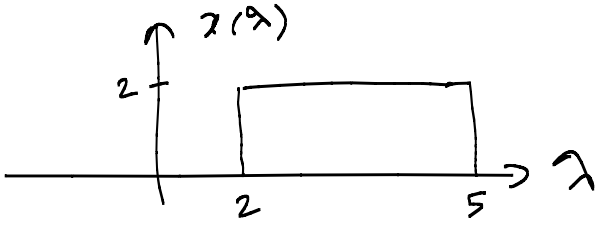
$f_1(t) * f_2(t) =$

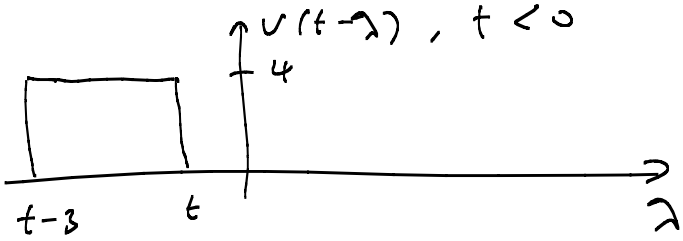


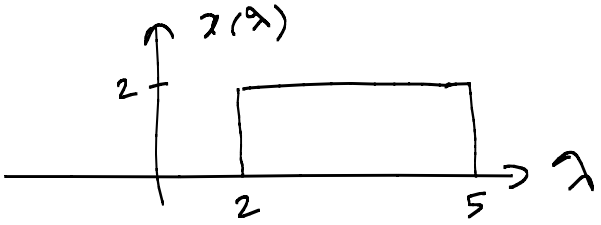
$x(t) * v(t) = ?$

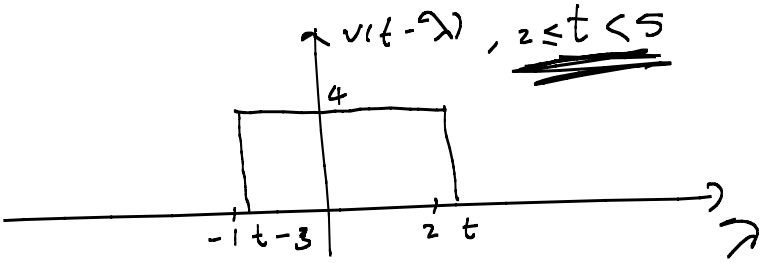

x(t)
2
2
5
t

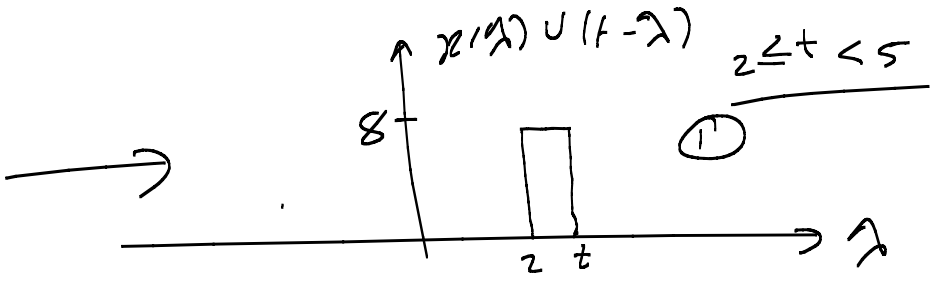

v(t)
4
3
t

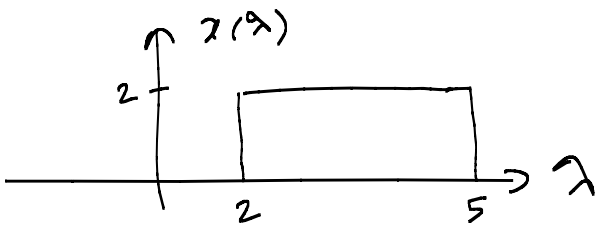

x(λ)
2
2
5
λ

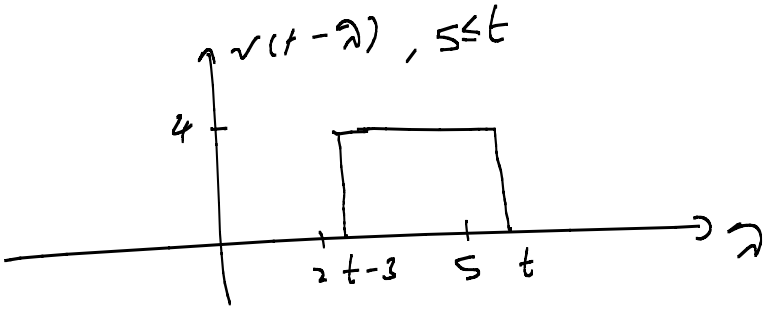

v(t-λ) , t < 0
4
t-3
t
λ

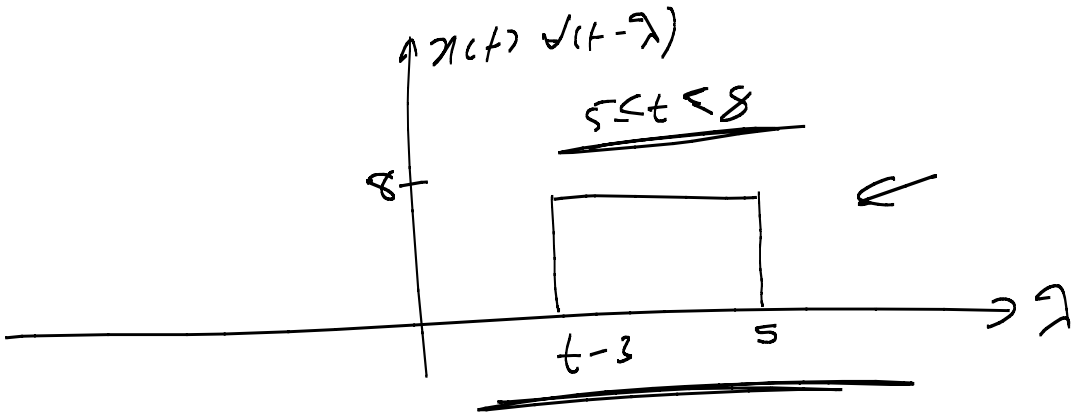

x(λ)
2
2
5
λ

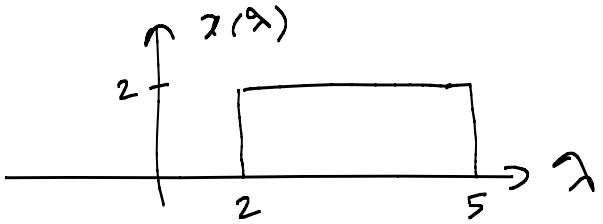

v(t-λ) , 2 ≤ t < 5
4
-1
t-3
2
t
λ

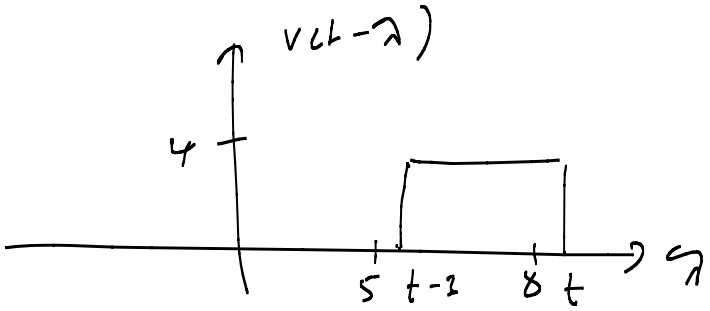

x(λ) v(t-λ)
2 ≤ t < 5
8
1
2
t
λ


x(λ)
2
2
5
λ


v(t-λ) , 5 ≤ t
4
2
t-3
5
t
λ


x(t) v(t-λ)
8 > t ≥ 5
8
t-3
5
λ





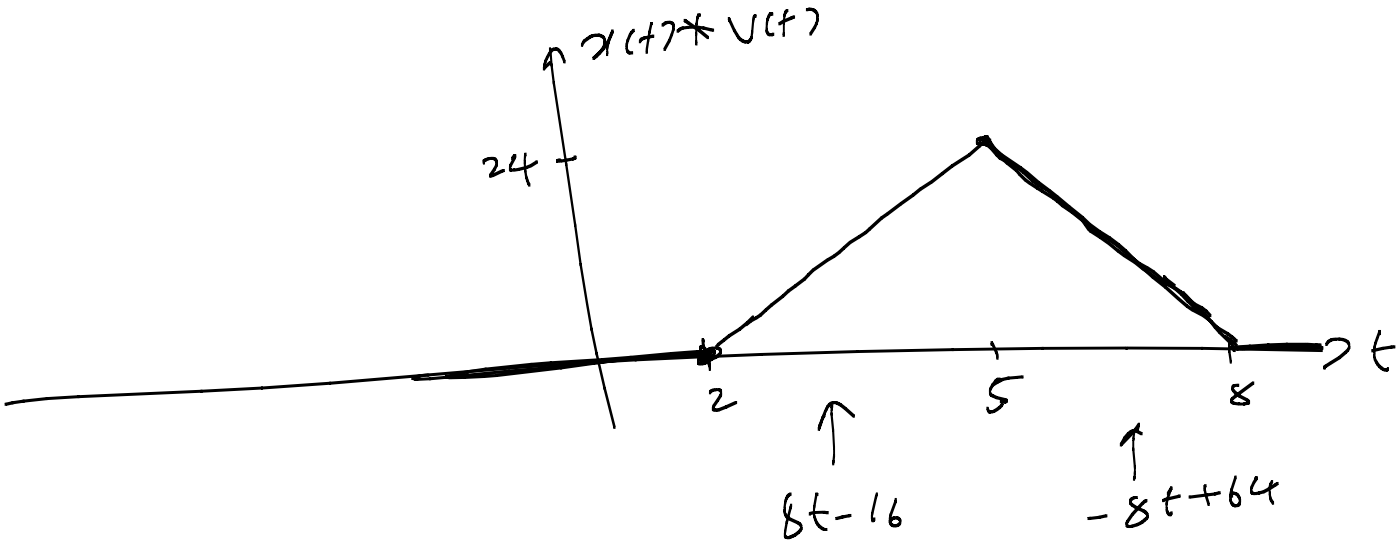
$x(t) * v(t) = 0$ for $t < 2$ & $t \geq 8$

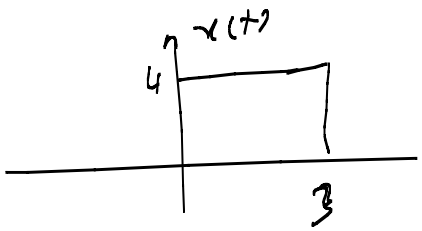
for $2 \leq t < 5$

$$x(t) * v(t) = \int_2^t 8 \, d\lambda = 8\lambda \Big|_2^t = \underline{8t - 16}$$

for $\underline{5 \leq t < 8}$ $\quad x(t) * v(t) = \int_{t-3}^{5} 8 \, d\lambda = 8\lambda \Big|_{t-3}^{5} = 8(5 - t + 3)$

$$= -8t + 64$$

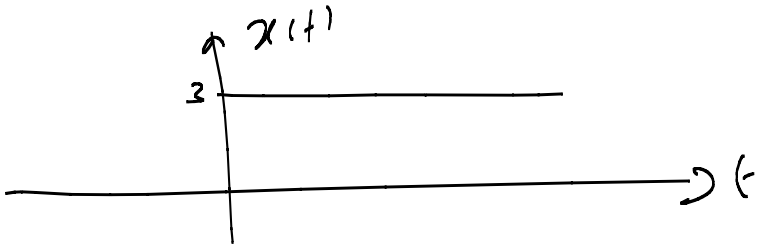


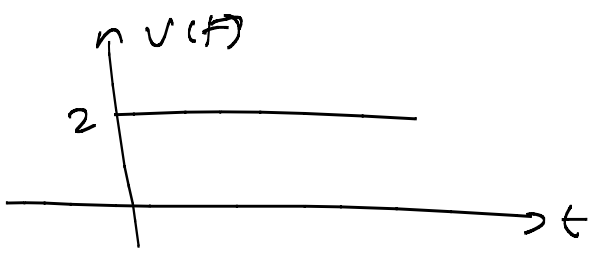




$x(t) = 4u(t) - 4u(t-3)$

$v(t) = 3u(t) - 3u(t-2)$





$$y[n+2] + 2y[n+1] + 3y[n] = 2x[n+1]$$

$$h[n+2] + 2h[n+1] + 3h[n] = 2\delta[n+1]$$

$$h[n+2] = -2h[n+1] - 3h[n] + 2\delta[n+1]$$

$$h[0] = -2h[-1] - 3h[-2] + 2\delta[-1] = 0$$

$$h[1] = -2h[0] - 3h[-1] + 2\delta[0] = 2$$

$$h[2] = -2h[1] - 3h[0] + 2\delta[1] = -4$$

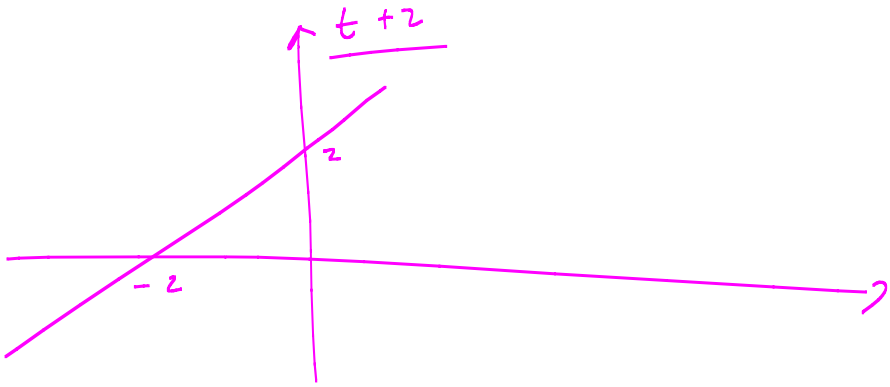
$$h[3] = -2h[2] - 3h[1] + 2\delta[2] = 2$$

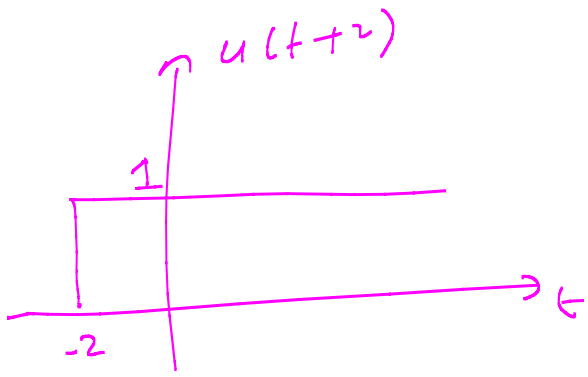
$$h[4] = -2h[3] - 3h[2] + 2\delta[3]$$

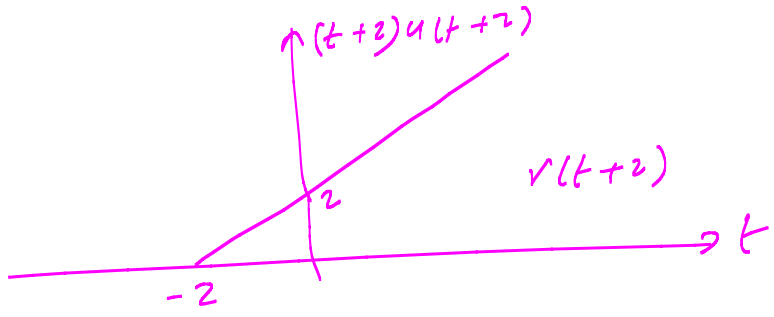
$$= 8$$

$$h[5] = \cdots$$

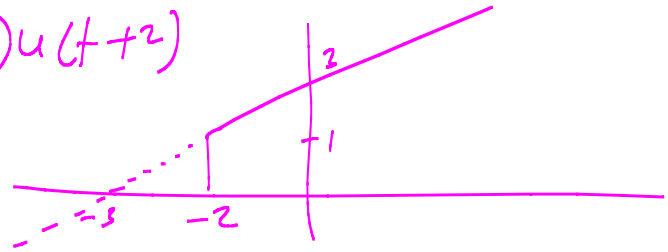
$$x(t) = (t+2)u(t+2) = r(t+2)$$

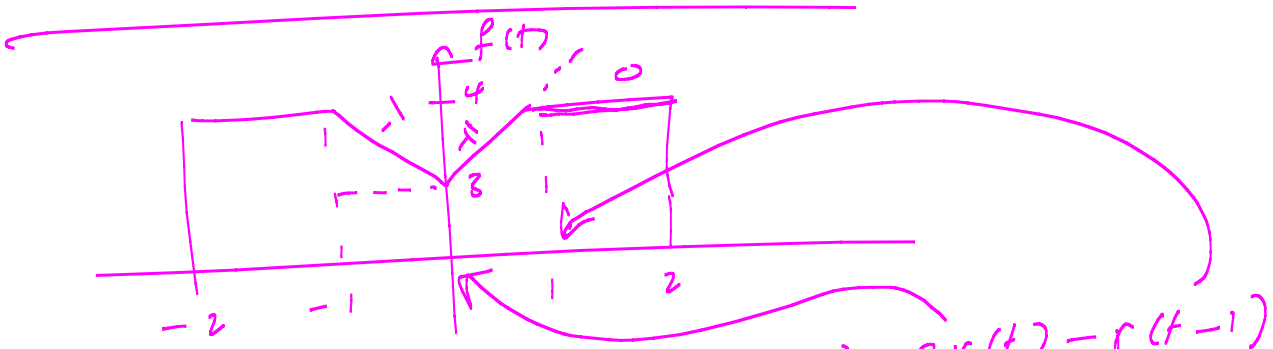




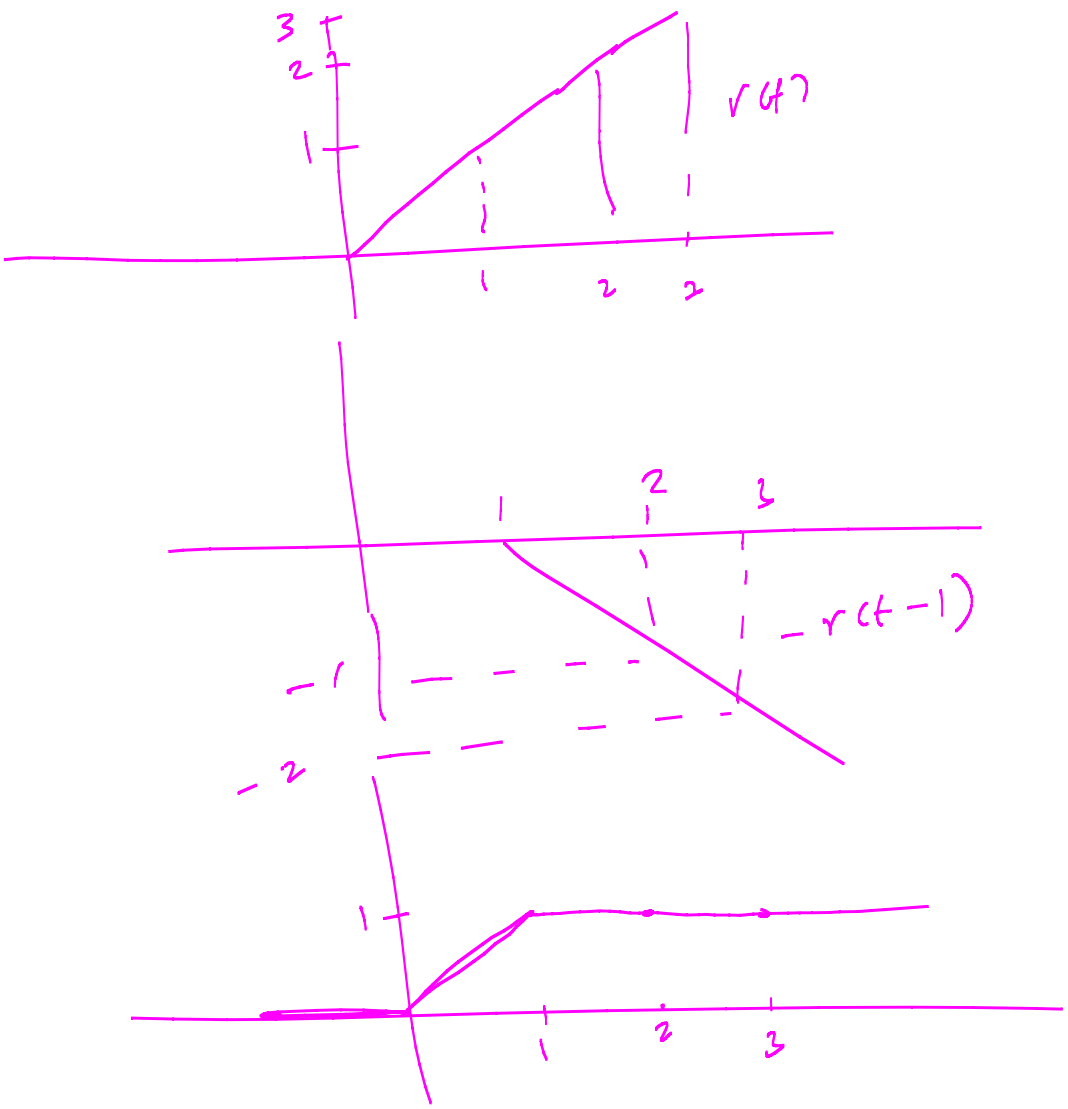


$(t+3)u(t+2)$





$$f(t) = 4u(t+2) - r(t+1) + 2r(t) - r(t-1) - 4u(t-2)$$


3
2
1
r(t)
1
2
3
2
3
-r(t-1)
-1
-2
1
2
3